2.4 EXACT EQUATIONS

$$M(x,y)\,dx + N(x,y)\,dy = 0$$ DIFFERENTIAL FORM

EQUIVALENT TO

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

AND

$$M(x,y)\frac{dx}{dy} + N(x,y) = 0$$

FROM MATH 1D:

IF $z = F(x,y)$

THEN $dz = \dfrac{\partial F}{\partial x}dx + \dfrac{\partial F}{\partial y}dy$ OR $dz = F_x\,dx + F_y\,dy$

TOTAL DIFFERENTIAL

WHAT IS THE TOTAL DIFFERENTIAL ALONG THE CURVE

$F(x,y) = c$?

CONTOUR LINE

$$F_x\,dx + F_y\,dy = 0$$

IE. $F(x,y) = c$ IS A SOL'N OF DE $F_x\,dx + F_y\,dy = 0$

FOR ALL $c \in \mathbb{R}$

SHOW THAT $\underbrace{\sin(xy)}_{F}=C$ IS A SOL'N OF

$$y\cos xy\,dx + x\cos xy\,dy = 0$$

$$F_x = y\cos(xy)$$
$$F_y = x\cos(xy)$$

SAME

IF $\sin xy = C$

THEN $y\cos xy\,dx + x\cos xy\,dy = 0$ $\quad (F_x dx + F_y dy = 0)$

SO, $\sin xy = C$ IS A SOL'N OF THE GIVEN DE

OR BY IMPLICIT DIFFERENTIATION

SAME

$$\frac{d}{dx}\sin xy = \frac{d}{dx}C$$

$$\underbrace{\frac{d}{d(xy)}\sin(xy)}\frac{d(xy)}{dx} = 0 \qquad \text{(CHAIN RULE)}$$

$$\cos(xy)\left(1\cdot y + x\cdot\frac{dy}{dx}\right) = 0$$

$$y\cos xy + (x\cos xy)\frac{dy}{dx} = 0$$

$$y\cos xy\,dx + x\cos xy\,dy = 0$$

DEF'N: $M(x,y)\,dx + N(x,y)\,dy = 0$

IS EXACT IF, FOR SOME FUNCTION $F(x,y)$,

$$M(x,y) = F_x(x,y)$$

$$\text{AND } N(x,y) = F_y(x,y)$$

IF THE EQUATION IS EXACT, THEN $F(x,y) = c$ IS A SOL'N OF THE DE FOR ALL $c \in \mathbb{R}$

CONSIDER $(2xy+3)\,dx + (x^2-1)\,dy = 0$

IS THE DE EXACT?

IE. IS THERE $F(x,y)$ SUCH THAT $F_x = 2xy+3$, $F_y = x^2-1$?

$$F_x = 2xy+3$$

$$\frac{\partial F}{\partial x} = 2xy+3$$

$$F = \int (2xy+3)\,dx$$

$$= x^2y + 3x + C(y)$$

$$F_y = x^2-1$$

$$\frac{\partial F}{\partial y} = x^2-1$$

$$F = \int (x^2-1)\,dy$$

$$= (x^2-1)y + K(x)$$

$$F(x,y) = x^2y + 3x + C(y) = (x^2-1)y + K(x)$$

$$x^2y + 3x + C(y) = x^2y + K(x) - y$$

$$C(y) = -y \quad \text{AND} \quad K(x) = 3x$$

$$\text{SO } F(x,y) = x^2y + 3x - y$$

$$(x^2-1)y + 3x$$
$$= x^2y + 3x - y$$

CHECK $F_x = 2xy + 3$

$F_y = x^2 - 1$

IF $F(x,y) = x^2y + 3x - y = C$

THEN $F_x\,dx + F_y\,dy = \underbrace{(2xy+3)\,dx + (x^2-1)\,dy = 0}$

SO, THIS DE IS EXACT

AND $F(x,y) = x^2y + 3x - y = C$ (OR JUST $x^2y + 3x - y = C$)

IS A SOL'N OF THE DE

OUR LAST METHOD REQUIRED US TO SOLVE DE
TO DETERMINE IF IT WAS EXACT

IF THE DE HAD NOT BEEN EXACT, WE HAD TO DO A LOT OF UNNECESSARY WORK

FROM MATH 1D:

$$(F_x)_y = (F_y)_x$$ UNDER CERTAIN CONTINUITY CONDITIONS

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$

SO, IF $M\,dx + N\,dy = 0$ IS EXACT, THEN $M_y = N_x$

$M = F_x$ AND $N = F_y$ FOR SOME F

$$M_y = (F_x)_y = (F_y)_x = N_x$$

AND ALSO IF $M_y = N_x$ THEN $M\,dx + N\,dy = 0$ IS EXACT

RECONSIDER $(2xy+3)\,dx + (x^2-1)\,dy = 0$

$$M = 2xy+3 \qquad N = x^2-1$$

$$M_y = 2x \quad = N_x = 2x$$

SO, THE DE IS EXACT

AND WE CAN SOLVE IT

$\downarrow$

$$F = \int M\,dx + C(y) = \int N\,dy + K(x)$$

COMPARE TO FIND $C(y), K(x)$

+ SUB BACK INTO F

SOLUTION OF DE IS

$$F(x,y) = c$$

IS $(xe^{xy} - 2y)\,dy + (ye^{xy} + 2x)\,dx = 0$ EXACT?

IF SO, SOLVE IT

$$\frac{\partial}{\partial x}(xe^{xy} - 2y) =$$

$$\frac{\partial}{\partial y}(ye^{xy} + 2x) =$$